Gauge Invariance and Stability of SNO vacuum in QCD

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We point out a critical defect in the calculation of the functional determinant of the gluon loop in the Savvidy-Nielsen-Olesen (SNO) effective action. We prove that the gauge invariance naturally exclude the unstable tachyonic modes from the gluon loop integral. This guarantees the stability of the magnetic condensation in QCD.

The confinement problem in quantum chromodynamics (QCD) is probably one of the most challenging problems in theoretical physics. It has long been argued that the confinement in QCD can be triggered by the monopole condensation [1, 2]. Indeed, if one assumes monopole condensation, one can easily argue that the ensuing dual Meissner effect could guarantee the confinement of color [3, 4]. But it has been extremely difficult to prove the monopole condensation in QCD.

A natural way to establish the monopole condensation in QCD is to demonstrate that the quantum fluctuation triggers a phase transition through the dimensional transmutation known as the Coleman-Weinberg mechanism [5]. To prove the monopole condensation, one need to demonstrate such a phase transition in QCD. There have been many attempts to do so with the one-loop effective action of QCD [6, 7, 8]. Savvidy has first calculated the effective action of $SU(2)$ QCD in the presence of an *ad hoc* color magnetic background, and has almost "proved" the magnetic condensation. In particular, he showed that the quantum effective potential obtained from the real part of the one-loop effective action has the minimum at a non-vanishing magnetic background [6]. Unfortunately, the calculation repeated by Nielsen and Olesen showed that the effective action contains an extra imaginary part which destablizes the magnetic condensation [7]. This instability of the "Savvidy-Nielsen-Olesen (SNO) vacuum" has destroyed the hope to establish the monopole condensation in QCD with the effective action [8].

Recently, however, there has been a new attempt to calculate the one-loop effective action of QCD with a gauge independent separation of the non-Abelian monopole background from the quantum field [9, 10]. Remarkably, in this calculation the effective action has been shown to produce no imaginary part in the presence of the monopole background, but a negative imaginary part in the presence of the pure color electric background. This implies that in QCD the non-Abelian monopole background produces a stable monopole condensation, but the color electric background becomes unstable by generating a pair annhilation of the valence gluon. The new result sharply contradicts with the earlier results, in particular on the stability of the monopole condensation. This has resurrected the old controversy on the stability of monopole condensation.

To resolve the controversy it is important to understand the origin of the instability of the SNO vacuum. The energy of a charged vector field moving around a constant magnetic field depends on the spin orientation of the vector field, and when the spin is anti-parallel to the magnetic field, the zeroth Landau level has a negative energy. Because of this the functional determinant of the gluon loop in the SNO magnetic background necessarily contains negative eigenvalues which create a severe infra-red divergence in the effective action. And, when one regularizes this divergence with the ζ-function regularization, one obtains the well-known imaginary component in the effective action which destablizes the magnetic condensation [7]. This tells that the instability of the SNO vacuum originates from the the negative eigenvalues of the functional determinant. Since the origin of the negative eigenvalues is so obvious, the instability of the SNO vacuum has become the prevailing view [7, 8].

This view, however, is not without defect. To see this notice that the eigenfuctions corresponding to the negative eigenvalues describes tachyons which are unphysical. This implies that one should exclude these tachyons in the calculation of the effective action, unless one wants to allow the violation of causality in QCD. Unfortunately the standard ζ-function regularization fails to remove the contribution of the tachyonic eigenstates because it is insensitive to causality. On the other hand, if we adopt the infra-red regularization which respects the causality, the resulting effective action has no imaginary part [9, 10]. But since the ζ-function regularization has worked so well
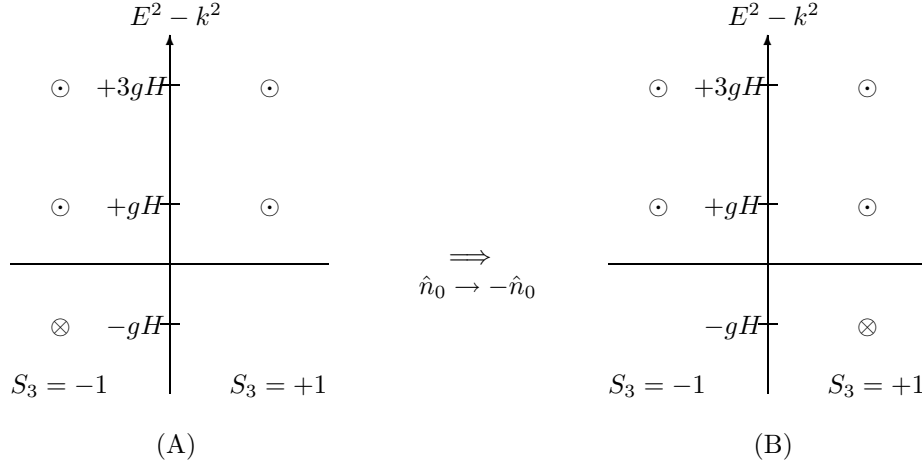
FIG. 1: The eigenvalues of the functional determinant of the gluon loop. When the gluon spin is anti-parallel to the magnetic field ($S_3 = -1$), the ground state (with $n = 0$) becomes tachyonic when $k^2 < gH$. Notice, however, that under the color reflection of $\hat{n}_0$ to $-\hat{n}_0$ H changes to $-H$ so that the eigenvalues change from (A) to (B). This shows that the spin polarization direction of gluon is a gauge artifact. This excludes the tachyons from the functional determinant.

in quantum field theory, there seems no compelling reason why it should not work in QCD. So we need to find an independent argument which can remove the negative eigenvalues from the functional determinant.

The purpose of this paper is to show that a proper implementation of the gauge invariance in the calculation of the functional determinant of the gluon loop excludes the unstable tachyonic modes, and thus naturally restore the stability of the magnetic background. This tells that it is the incorrect calculation of the functional determinant, not the ζ-function regularization, which causes the instability of the SNO vacuum. This means that tachyons should not have been there in the first place. They were there to create a mirage, not physical states.

In the old approach Savvidy starts from the SNO background which is not gauge invariant [6, 7]. Because of this the functional determinant of the gluon loop contains the tachyonic eigenstates when the gluon spin is anti-parallel to the magnetic field. To cure this defect Nielsen and Olesen has introduced the gauge invariant "Copenhagen vacuum" [7]. Although conceptually appealing, however, this Copenhagen vacuum was not so useful in proving the stability of the monopole condensation. In the following we show that the gauge invariant functional determinant should not depend on the spin polarization of the gluon. This tells that, if we impose the gauge invariance properly, the instability of the SNO background should disappear.

To obtain the one-loop effective action one must divide the gauge potential $\vec{A}_\mu$ into the slow-varying classical background $\vec{B}_\mu$ and the fluctuating quantum part $\vec{Q}_\mu$,

$$\vec{A}_\mu = \vec{B}_\mu + \vec{Q}_\mu, \tag{1}$$

and integrate out the quantum part. The gluon loop and the ghost loop integrals give the following functional determinants [6, 7, 9, 10]

$$\mathrm{Det}^{-\frac{1}{2}} K_{\mu\nu}^{ab} = \mathrm{Det}^{-\frac{1}{2}}\Big(-g_{\mu\nu}\bar{D}_{ab}^2 - 2g\epsilon_{abc}G_{\mu\nu}^c \Big),$$
$$\mathrm{Det}\, M_{ab} = \mathrm{Det}\Big(-\bar{D}_{ab}^2 \Big), \tag{2}$$

where $\bar{D}_\mu$ and $\vec{G}_{\mu\nu}$ are the covariant derivative and field strength of the background $\vec{B}_\mu$. From this one has

$$\Delta S = \frac{i}{2} \ln \mathrm{Det} K - i \ln \mathrm{Det} M. \tag{3}$$

Savvidy has chosen a covariantly constant color magnetic field as the classical background [6, 7, 8]

$$\vec{B}_\mu = \frac{1}{2} H_{\mu\nu} x_\nu \hat{n}_0, \qquad \vec{G}_{\mu\nu} = H_{\mu\nu}\hat{n}_0,$$
$$\bar{D}_\mu \vec{G}_{\mu\nu} = 0, \tag{4}$$

where $H_{\mu\nu}$ is a constant magnetic field and $\hat{n}_0$ is a constant unit isovector. The calculation of the functional determinant (2) amounts to the calculation of the energy eigenvalues of a massless charged vector field (the valence gluon) in a constant external magnetic field $H_{\mu\nu}$ [7]. Choosing the direction of the magnetic field to be the z-direction, one obtains the well-known eigenvalues

$$E^2 = 2gH(n + \frac{1}{2}) + k^2 \pm 2gH,$$
$$H = H_{12}, \tag{5}$$

where k is the momentum of the eigen-function in z-direction. Notice that the $\pm$ signature correspond to the spin $S_3 = \pm 1$ of the valence gluon. So, when $n = 0$, the eigen-functions with $S_3 = -1$ have an imaginary energy
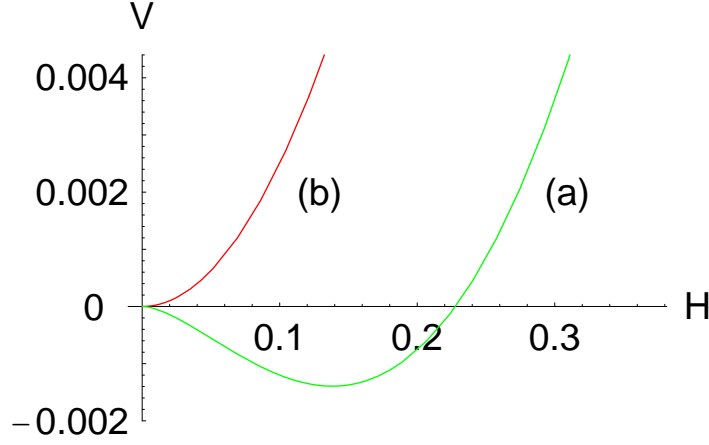
FIG. 2: The effective potential of SU(2) QCD in the pure magnetic background. Here (a) is the effective potential and (b) is the classical potential.

when $k^2 < gH$, and thus become tachyons which violate the causality.

From (5) one obtains

$$\Delta S = i \ln \mathrm{Det} \Big[(-\bar{D}^2 + 2gH)(-\bar{D}^2 - 2gH) \Big], \quad (6)$$

so that [7, 8]

$$\mathcal{L}_{eff} = -\frac{H^2}{4} - \frac{11g^2}{96\pi^2} H^2 (\ln \frac{gH}{\mu^2} - c) + i \frac{g^2}{16\pi} H^2, \quad (7)$$

where c is a regularization-dependent connstant. This contains the well-known imaginary part which destablizes the SNO vacuum [7, 8].

Notice, however, that the background $\vec{G}_{\mu\nu}$ must be gauge covariant. So one can change $\vec{G}_{\mu\nu}$ to $-\vec{G}_{\mu\nu}$, and thus $H_{\mu\nu}$ to $-H_{\mu\nu}$, by a gauge transformation (with the color reflection of $\hat{n}_0$ to $-\hat{n}_0$) so that they are gauge equivalent. *This tells that the polarization direction of the magnetic background is a gauge artifact. Furthermore, under this gauge transformation the eigenvalues of $S_3 = +1$ ($S_3 = -1$) shift nagatively (positively) by a factor $2gH$. And obviously only the eigenvalues which are invariant under this transformation should qualify to be gauge invariant. This means that the gauge invariant eigenstates are those which are independent of the spin orientation of the valence gluon which appear in both $S_3 = +1$ and $S_3 = -1$ simultaneously. In particular, this tells that the tachyonic eigenstates are not gauge invariant.* This is shown schematically in Fig. 1, where (A) transforms to (B) (and vise versa) under the color reflection. This tells that one must exclude the tachyons in one's calculation of the effective action.

If one does so, the effective action (6) changes to

$$\Delta S = i \ln \mathrm{Det} \Big[(-\bar{D}^2 + 2gH)(-\bar{D}^2 + 2gH) \Big], \quad (8)$$

which has no infra-red divergence at all. This precludes the necessity to make any infra-red regularization. From this we have [9, 10]

$$\mathcal{L}_{eff} = -\frac{H^2}{4} - \frac{11g^2}{96\pi^2} H^2 (\ln \frac{gH}{\mu^2} - c), \quad (9)$$

which clearly has no imaginary part.

A simple way to understand the above result is to remember that the effective action is nothing but the vacuum to vacuum amplitude in the presence of the classical background,

$$\exp \Big[i S_{eff}(\vec{B}_\mu) \Big] = < \Omega_+ | \, \Omega_- > \Big|_{\vec{B}_\mu}$$
$$= \sum_{|n_i>} < \Omega_+ | \, n_i >< n_i \, | \, \Omega_- > \Big|_{\vec{B}_\mu}, \quad (10)$$

where $|\Omega>$ is the vacuum and $|n_i>$ is a complete set of orthonormal states of QCD. In this view the gluon loop integral corresponds to the summation of the complete set of states. And obviously the complete set should not include the tachyons, unless one wants to assert that the physical spectrum of QCD must contain the unphysical states which violate the causality and the gauge invariance. This justifies the exclusion of the tachyons in the calculation of the functional determinant.

The effective action (9) generates the much desired dimensional transmutation in QCD [9, 10]. To demonstrate this notice that the effective action provides the following effective potential

$$V = \frac{H^2}{4} \Big[1 + \frac{11g^2}{24\pi^2} (\ln \frac{gH}{\mu^2} - c) \Big]. \quad (11)$$

So we define the running coupling $\bar{g}$ by [6, 9]

$$\frac{\partial^2 V}{\partial H^2} \Big|_{H=\bar{\mu}^2} = \frac{1}{2} \frac{g^2}{\bar{g}^2}. \quad (12)$$

With the definition we find

$$\frac{1}{\bar{g}^2} = \frac{1}{g^2} + \frac{11}{24\pi^2}(\ln\frac{g\bar{\mu}^2}{\mu^2} - c + \frac{3}{2}), \qquad (13)$$

from which we obtain the following β-function

$$\beta(\bar{\mu}) = \bar{\mu}\frac{\partial\bar{g}}{\partial\bar{\mu}} = -\frac{11\bar{g}^3}{24\pi^2}. \qquad (14)$$

This is exactly the same β-function that one obtained from the perturbative QCD to prove the asymptotic freedom [11].

In terms of the running coupling the renormalized potential is given by

$$V_{\rm ren} = \frac{H^2}{4}\left[1 + \frac{11\bar{g}^2}{24\pi^2}(\ln\frac{H}{\bar{\mu}^2} - \frac{3}{2})\right], \qquad (15)$$

which generates a non-trivial local minimum at

$$< H > = \frac{\bar{\mu}^2}{\bar{g}}\exp\left(-\frac{24\pi^2}{11\bar{g}^2} + 1\right). \qquad (16)$$

Notice that with $\bar{\alpha}_s = 1$ we have

$$\frac{< H >}{\bar{\mu}^2} = 0.13819.... \qquad (17)$$

This is nothing but the desired magnetic condensation. The corresponding effective potential is plotted in Fig. 2, where we have assumed $\bar{\alpha}_s = 1$ and $\bar{\mu} = 1$.

Nielsen and Olesen have suggested that the existence of the unstable tachyonic modes are closely related with the asymptotic freedom in QCD [7]. Our analysis tells that this need not be true. Obviously our asymptotic freedom follows from a stable monopole condensation.

To establish the monopole condensation in QCD with the effective action has been extremely difficult to attain. The central issue here has been the stability of the monopole condensation. The earlier attempts to prove the monopole condensation have produced a negative result, because the SNO background is not gauge invariant [6, 7]. In this paper we have shown that a proper implementation of gauge invariance naturally restores the stability of the magnetic condensation.

It is not surprising that the gauge invariance plays the crucial role in the stability of the monopole condensation. From the beginning the gauge invariance has been the main motivation for the confinement in QCD. It is this gauge invariance which forbids colored objects from the physical spectrum of QCD. This necessitates the confinement of color. So it is only natural that the gauge invariance assures the stability of the monopole condensation, and thus the confinement of color.

Finally it must be emphasized that there are actually two ways to exclude the unphysical modes, when one calculates the functional determinant or when one makes the infra-red regularization. We have already shown how to do this when we make the infra-red regularization which respects the causality [9, 10]. In this paper we have shown how to do this when we calculate the functional determinant properly implementing the gauge invariance. And they produce the same effective action. It is really remarkable that two completely independent principles, the causality and the gauge invariance, both endorse the stability of the magnetic condensation in QCD.

The detailed discussion of the above result will be published elsewhere [12].

Acknowledgements

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